Solid Mechanics: Non-linear elasticity

In non-linear elasticity we study the constitutive response of materials
When the displacement and deformation are not small.

We aim to model materials such as elastomers and some biological tissues

From the book: Mechanics of Continuous Media: an Introduction
J Botsis and M Deville, PPUR 2018.
Solutions: https://www.epflpress.org/produit/908/9782889152810/mechanics-of-continuous-media

Solid Mechanics: Non-linear elasticity

Positive definite tensor

It satifies the following relation

$$\forall \boldsymbol{v} \in E^3, \quad \boldsymbol{v} \cdot \boldsymbol{L v} > 0$$

It can be shown that the eigenvalues of a positive Definite tensor are all positive:

For the tensor $\boldsymbol{L}$ with one of its eigenvalue λ and and corresponding eigenvector $\boldsymbol{n}$, we can easily see that since

$$\boldsymbol{L n} = \lambda \boldsymbol{n} \Longrightarrow \boldsymbol{n} \cdot \boldsymbol{L n} = \lambda > 0$$

Spectral decomposition of a tensor or spetral represenation of a tensor

For a tensor $\boldsymbol{L}$ with eigenvalues λ_1, λ_2, λ_3, and corresponding eigenvectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$.
The orthogonal eigenvectors form a basis for the spectral decompostion writen as follows:

$$\boldsymbol{L} = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

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Theorem (square root)

For a symmetric, positive definite tensor $\boldsymbol{C}$ with eigenvalues λ_i^2 and corresponding eigenvectors $\boldsymbol{n}_i$, there exists a symmetric positive definite tensor $\boldsymbol{U}$ such that:

$$\boldsymbol{U}^2 = \boldsymbol{C}$$

and denote it as $\sqrt{\boldsymbol{C}} = \boldsymbol{U}$

These two tensors have the following spectral form:

$$\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \, \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

Theorem (polar decomposition)

For a tensor $\boldsymbol{F}$ with determinant det $\boldsymbol{F} > 0$ there exist symmetric positive definite tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ and a rotation (an orthogonal tensor with a positive Determinant equal to 1) $\boldsymbol{R}$ such that:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$$

These decompositions are unique and we have:

$$\boldsymbol{U} = \sqrt{\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}} \quad \text{and} \quad \boldsymbol{V} = \sqrt{\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}}$$

Representation $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$ is called right decomposition.

Representation $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}$ is called left decomposition.

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$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

$$\boldsymbol{U}^2 = \boldsymbol{U}\boldsymbol{U} =$$

$$= \sum_{i=1}^{3} \lambda_i^2 (\boldsymbol{n}_i \otimes \boldsymbol{n}_i)(\boldsymbol{n}_i \otimes \boldsymbol{n}_i) =$$

$$= \sum_{i=1}^{3} \lambda_i^2 (\boldsymbol{n}_i \otimes \boldsymbol{n}_i) = \boldsymbol{C}$$

Note that

$$(\boldsymbol{n}_i \otimes \boldsymbol{n}_i)(\boldsymbol{n}_j \otimes \boldsymbol{n}_j) = \begin{cases} 0 & \text{if } i \neq j \\ (\boldsymbol{n}_i \otimes \boldsymbol{n}_i) & \text{if } i = j \, . \end{cases}$$

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Isotropic tensor function of a symmetric tensor

By definition an tensor isotropic function $\boldsymbol{f}$, for which the variable is a 2nd order symmetric tensor $\boldsymbol{T}$, satisfies the identity:

$$\boldsymbol{Q}\boldsymbol{f}\,(\boldsymbol{T})\boldsymbol{Q}^T = \boldsymbol{f}(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T)$$

for any orthogonal tensor $\boldsymbol{Q}$.

For a symmetric tensor $\boldsymbol{L}$ the following relation is true:

$$\boldsymbol{L} = \boldsymbol{f}\,(\boldsymbol{T})$$

Rivlin-Ericksen representation Theorem

The last expression can be written in the form

$$\begin{aligned}\boldsymbol{L} = \varphi_0(I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T}))\boldsymbol{I} + \varphi_1(I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T}))\boldsymbol{T} \\ + \varphi_2(I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T}))\boldsymbol{T}^2\,,\end{aligned}$$

$\varphi_i\ \ (i = 0,\ 1,\ 2)$ *are scalar functions of the invariants of* $\boldsymbol{T}$

Scalar function of a tensor

The function $\mathcal{W}(\boldsymbol{T})$ is defined as a scalar function of the tensor $\boldsymbol{T}$ and yield a scalar. When $\boldsymbol{T}$ is symmetric and the condition:

$$\mathcal{W}(\boldsymbol{T}) = \mathcal{W}(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T)$$

is satisfied, then $\mathcal{W}(\boldsymbol{T})$ is isotropic of $\boldsymbol{T}$ and is represented by:

$$\mathcal{W}(\boldsymbol{T}) = \Phi(I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T}))$$

where the parameters $I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T})$ are the invariants of $\boldsymbol{T}$. This is also equivalent to:

$$\mathcal{W}(\boldsymbol{T}) = \phi(\lambda_1, \lambda_2, \lambda_3)$$

where λ_1, λ_2, λ_3, are the eigenvalues of $\boldsymbol{T}$.

It can be shown that for the isotropic function $\mathcal{W}(\boldsymbol{T})$ its derivative with respect to $\boldsymbol{T}$ is is a tensor and given by:

$$\frac{\partial \mathcal{W}}{\partial \boldsymbol{T}} = \sum_{i=1}^{3} \frac{\partial \mathcal{W}}{\partial \lambda_i} \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

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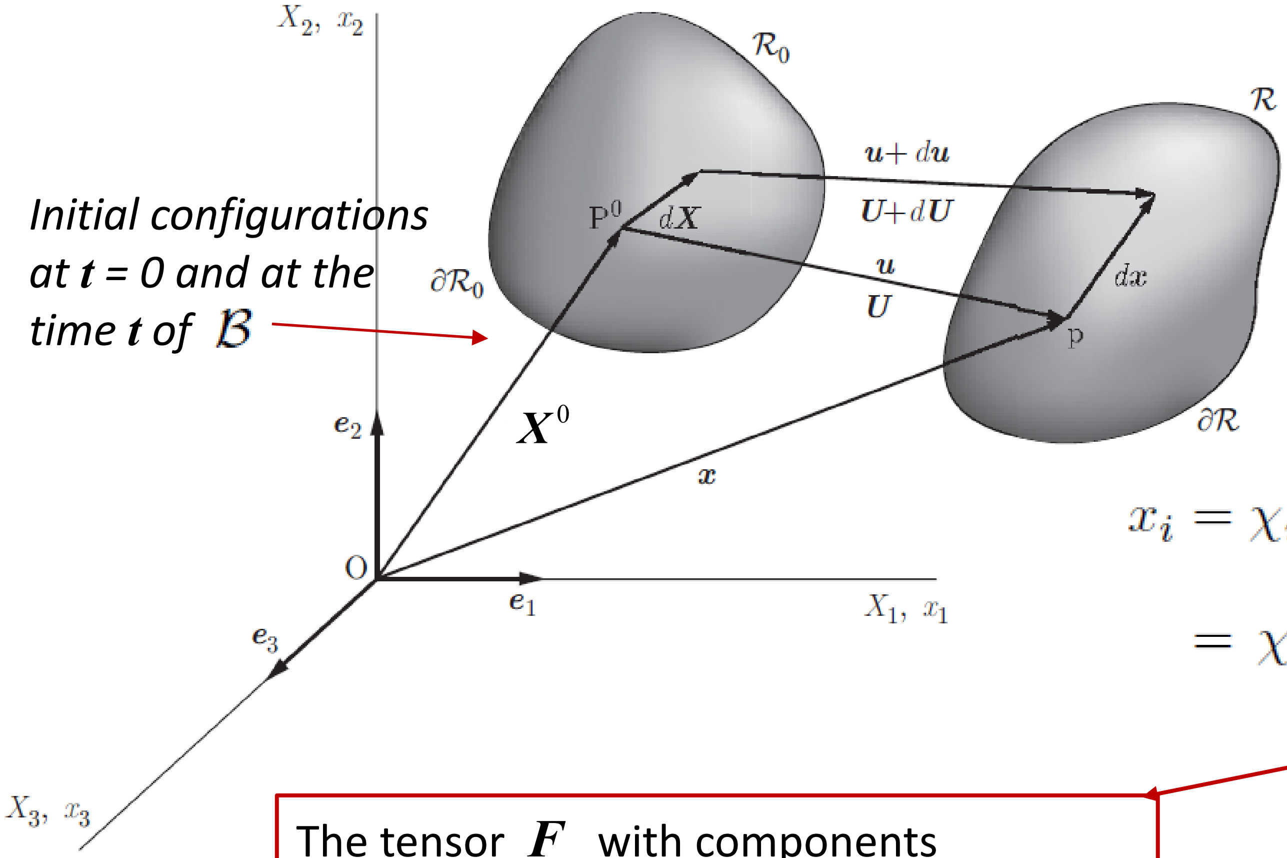
Deformation gradient tensor



*Initial configurations at **t** = 0 and at the time **t** of* $\mathcal{B}$

We consider a particle in configuration $\mathcal{R}_0$ with position $\boldsymbol{X}^0$ and a small neighborhood around it $\mathcal{V}$. .

Its motion is given by, $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$

For a sufficiently small $\mathcal{V}$. the motion for each particle in $\mathcal{V}$ is approximated by a Taylor series around $\boldsymbol{X}^0$ as follows

$$\begin{aligned} x_i &= \chi_i(X_k, t) \\ &= \chi_i(X_k^0, t) + \left.\frac{\partial \chi_i}{\partial X_j}\right|_{X_k^0} (X_j - X_j^0) + O(\|\boldsymbol{X} - \boldsymbol{X}^0\|^2) \end{aligned}$$

The tensor $\boldsymbol{F}$ with components

$$F_{ij} = \frac{\partial \chi_i}{\partial X_j}$$

is called the <u>Deformation gradient tensor .</u>

where

$$O\left(\|\boldsymbol{X} - \boldsymbol{X}^0\|^2\right) \sim C\|\boldsymbol{X} - \boldsymbol{X}^0\|^2 + \cdots$$

and C being a bounded constant.

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Deformation gradient tensor

If $\|\boldsymbol{X} - \boldsymbol{X}^0\|$ << 1

$$x_i = \chi_i(X_k, t)$$

$$= \chi_i(X_k^0, t) + \left.\frac{\partial \chi_i}{\partial X_j}\right|_{X_k^0} (X_j - X_j^0) + O(\|\boldsymbol{X} - \boldsymbol{X}^0\|^2)$$

→ $x_i \cong x_i^0 + F_{ij}(X_j - X_j^0)$ with $x_i^0 = \chi_i(X_k^0, t)$

or $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$

and for simplicity $F_{ij} = \dfrac{\partial x_i}{\partial X_j}$

Two assure the continuity of the material and the existence of continuous derivative the Jacobian J of $\boldsymbol{F}$ defined as:

$$J = \det\left(\frac{\partial \chi_i}{\partial X_j}\right) = \det \boldsymbol{F}$$

should satisfy the condition:

$$0 < J < \infty$$

which ensures the existence of the inverse $\boldsymbol{F}^{-1}$ of $\boldsymbol{F}$ with det $\boldsymbol{F}$ = 1/ J.

we use

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{U}(\boldsymbol{X}, t)$$

$$\boldsymbol{x} = \chi^{-1}(\boldsymbol{x}, t) + \boldsymbol{u}(\boldsymbol{x}, t)$$

$$\boxed{F_{ij} = \frac{\partial x_i}{\partial X_j}}$$

to obtain

$$F_{ij} = \delta_{ij} + \frac{\partial U_i}{\partial X_j}$$

$$F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j}$$

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Deformation tensors

Using $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$ in index form we have:

$$dx_i = F_{ij}\, dX_j$$

We can define the square ds o the vector $d\boldsymbol{x}$ as

$$ds^2 = \|d\boldsymbol{x}\|^2 = dx_m\, dx_m = F_{mi} F_{mj}\, dX_i\, dX_j$$

From this expression we can define the following tensor:

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \left(\boldsymbol{F}^T \boldsymbol{F}\right)^T \qquad C_{ij} = F_{mi} F_{mj}$$

Which is defined as the symmetric
right Cauchy-Green deformation tensor.

It is a **metric tensor** in that it can me used to calculate the length of $d\boldsymbol{x}$ as a function of the components $d\boldsymbol{X}$.

We can also calculate $d\boldsymbol{X}$ in terms of $d\boldsymbol{x}$ as follows:

with:

$$dX = F^{-1} dx \;\; ; \;\; dX_m = F^{-1}_{mi} dx_i$$

⬇

$$dS^2 = \|d\boldsymbol{X}\|^2 = dX_m\, dX_m = F^{-1}_{mi} F^{-1}_{mj}\, dx_i\, dx_j$$

and $F^{-1}_{mi} F^{-1}_{mj} = \left(F^T\right)^{-1}_{im} F^{-1}_{mj}$

we define the tensor:

$$\boldsymbol{c}^{-1} = \boldsymbol{F}^{-T} \boldsymbol{F}^{-1} = \left(\boldsymbol{F}^{-T} \boldsymbol{F}^{-1}\right)^T$$

or $c^{-1}_{ij} = F^{-1}_{mi} F^{-1}_{mj}$;

is the inverse of the **symmetric left Cauchy-Green deformation tensor** $\boldsymbol{c}$.

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Consider the tensor $\boldsymbol{U}$ with eigenvalues $\lambda_i \ (i = 1, 2, 3)$ and eigenvectors $\boldsymbol{A}_i$

$$\boldsymbol{U}\boldsymbol{A}_i = \lambda_i \boldsymbol{A}_i \qquad (\text{no sum over } i)$$

$\boldsymbol{U}$ is symmetric and positive definite

the λ_i are real and $\lambda_i > 0$

$$\boldsymbol{U} = \lambda_1 \boldsymbol{A}_1 \otimes \boldsymbol{A}_1 + \lambda_2 \boldsymbol{A}_2 \otimes \boldsymbol{A}_2 + \lambda_3 \boldsymbol{A}_3 \otimes \boldsymbol{A}_3$$

with $\quad \boldsymbol{A}_i \cdot \boldsymbol{A}_j = \delta_{ij}$

Using the relation

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \boldsymbol{U}\boldsymbol{R}^T \boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}^2$$

$$\Downarrow$$

$$\boldsymbol{C}\boldsymbol{A}_i = \lambda_i^2 \boldsymbol{A}_i \qquad (\text{no sum over } i)$$

$$\Downarrow$$

The deformation tensor $\boldsymbol{C}$ has λ_i^2 for eigenvalues and $\boldsymbol{A}_i \ (i = 1, 2, 3)$ for eigenvectors

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Using the polar decomposition theorem, we can express the deformation gradient tensor $\boldsymbol{F}$ as follows:

$$\underbrace{\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}}_{\text{right polar decomposition}} = \underbrace{\boldsymbol{V}\boldsymbol{R}}_{\text{left polar decomposition}}$$

right polar decomposition

left polar decomposition

The three tensors are unique

$\boldsymbol{R}$ expresses a rotation; $\boldsymbol{U}$ and $\boldsymbol{V}$ are called the right and left stretch tensors:

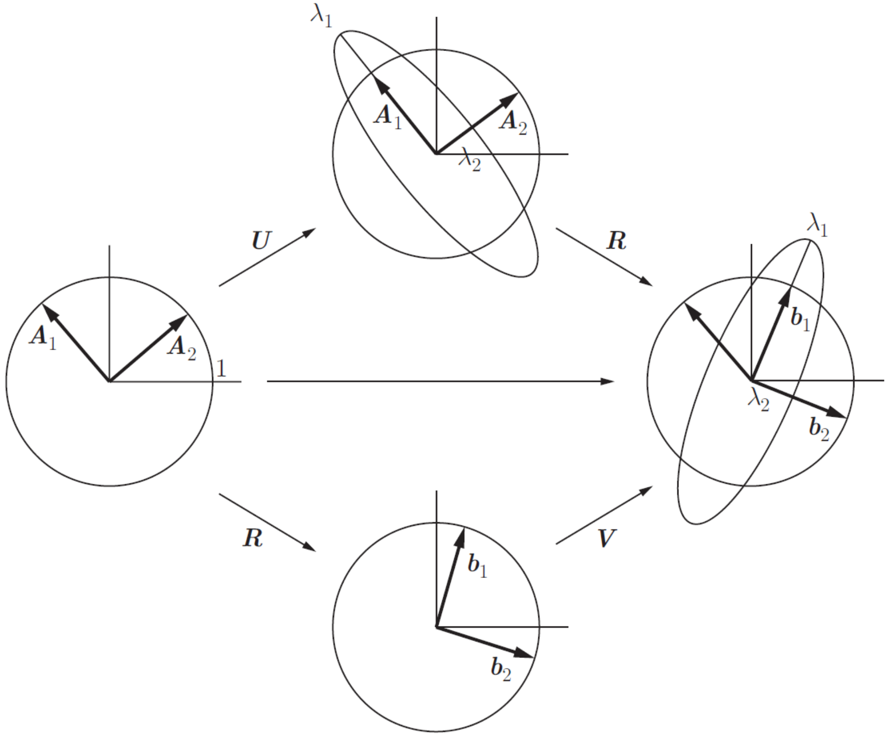
when $\boldsymbol{R} = \boldsymbol{I} \Rightarrow \boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V}$

we have pure deformation.

GEOMETRIC INTERPRETATION

$$d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X} \Rightarrow d\boldsymbol{x} = \boldsymbol{R}\boldsymbol{U}\, d\boldsymbol{X}$$

the configurational change in the neighborhood of the material particle is obtained by the transformation of vector $d\boldsymbol{X}$ to a vector $\boldsymbol{U}d\boldsymbol{X}$ by a pure deformation $\boldsymbol{U}$ followed by a local rotation $\boldsymbol{R}$.



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Deformation tensors

The deformation tensors can also be expressed in terms of tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ by applying the polar decomposition:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$$

1: the right Cauchy-Green deformation tensor:

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}^2$$

2: the left Cauchy-Green deformation tensor:

$$\boldsymbol{c} = \boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{V}\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{V}^T = \boldsymbol{V}^2$$

$$\boldsymbol{c}^{-1} = \boldsymbol{F}^{-T}\boldsymbol{F}^{-1} = \mathbf{V}^{-2} \;;$$

3: the Green-Lagrange strain tensor:

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{I}\right)$$

4: the Euler-Almansi strain tensor:

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{c}^{-1}\right) = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{V}^{-2}\right)$$

Note that the rotation $\boldsymbol{R}$ does not affect the deformation and strain tensors.
(very important in continuum mechanics)

Also when $\boldsymbol{F} = \boldsymbol{Q}$ it is a rigid body motion:

From $F = RU$ we have:

$$Q = RU \quad \Rightarrow \quad R^{-1}Q = R^{-1}RU \quad \Rightarrow \quad R^{-1}Q = IU$$

without loss of generality we set:

$$Q = R$$

Because they are both orthogonal tensors

$$\Rightarrow \quad U = I \quad \Rightarrow \quad E = 0$$

and similarly $V = I \quad \Rightarrow \quad e = 0\,.$

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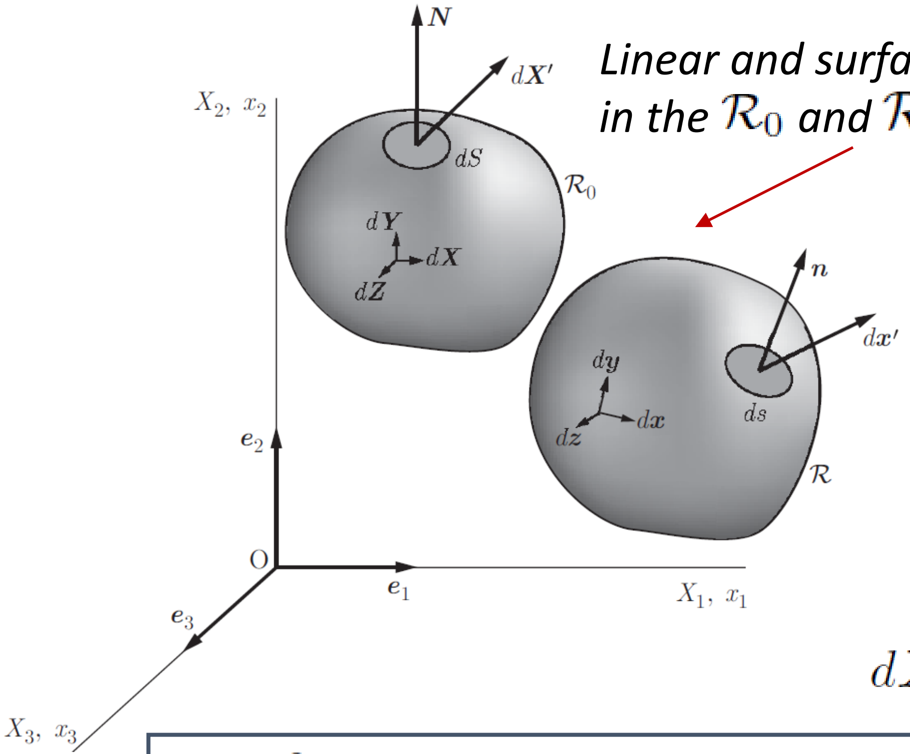

Linear and surface elements in the $\mathcal{R}_0$ and $\mathcal{R}$ of the $\mathcal{B}$

Description of a linear element in two configurations

Using $\boldsymbol{F}$ and the deformation tensors we can express The change in length of a linear element:

A linear element $d\boldsymbol{X}$ in the reference configuration has a norm:

$$\|d\boldsymbol{X}\| = (d\boldsymbol{X} \cdot d\boldsymbol{X})^{1/2}$$

After the motion $\boldsymbol{x} = \chi(\boldsymbol{X}, t)$ it becomes the element $d\boldsymbol{x}$ with norm:

$$\|d\boldsymbol{x}\| = (d\boldsymbol{x} \cdot d\boldsymbol{x})^{1/2}$$

Using $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$ and $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ we obtain:

$$\frac{\|d\boldsymbol{x}\|^2}{\|d\boldsymbol{X}\|^2} = \frac{\boldsymbol{F}\, d\boldsymbol{X} \cdot \boldsymbol{F}\, d\boldsymbol{X}}{\|d\boldsymbol{X}\|^2} = \\ = \frac{d\boldsymbol{X} \cdot \boldsymbol{F}^T \boldsymbol{F}\, d\boldsymbol{X}}{\|d\boldsymbol{X}\|^2} = \frac{d\boldsymbol{X} \cdot \boldsymbol{C}\, d\boldsymbol{X}}{\|d\boldsymbol{X}\|^2} .$$

$$d\boldsymbol{X} = \boldsymbol{N}\|d\boldsymbol{X}\|$$

$$\frac{\|d\boldsymbol{x}\|^2}{\|d\boldsymbol{X}\|^2} = \frac{d\boldsymbol{X} \cdot \boldsymbol{C}\, d\boldsymbol{X}}{\|d\boldsymbol{X}\|\,\|d\boldsymbol{X}\|} = \boldsymbol{N} \cdot \boldsymbol{C}\boldsymbol{N} = \lambda_N^2$$

λ_N is the stretch ratio at $\boldsymbol{X}$ in the direction $\boldsymbol{N}$

$$C = U^2$$

$$\frac{\|d\boldsymbol{x}\|}{\|d\boldsymbol{X}\|} = (\boldsymbol{N} \cdot \boldsymbol{U}^2 \boldsymbol{N})^{1/2} = (\boldsymbol{U}\boldsymbol{N} \cdot \boldsymbol{U}\boldsymbol{N})^{1/2} = \\ = \|\boldsymbol{U}\boldsymbol{N}\| = \lambda_N .$$

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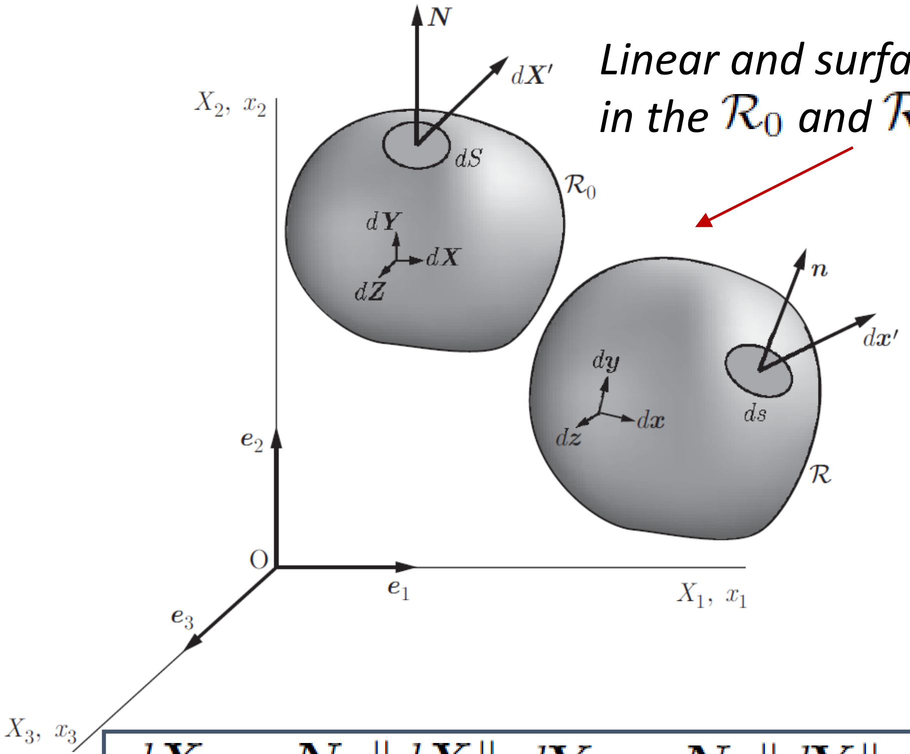

Linear and surface elements in the $\mathcal{R}_0$ and $\mathcal{R}$ of the $\mathcal{B}$

$$\boldsymbol{dX} = \boldsymbol{N}_x\|\boldsymbol{dX}\| \quad \boldsymbol{dY} = \boldsymbol{N}_y\|\boldsymbol{dY}\|$$

$\boldsymbol{N}_x$ and $\boldsymbol{N}_y$ are unit vectors along $\boldsymbol{X},\ \boldsymbol{Y}$

$$\|\boldsymbol{F}\,d\boldsymbol{X}\| = (\boldsymbol{F}\,d\boldsymbol{X}\cdot\boldsymbol{F}\,d\boldsymbol{X})^{1/2} = (d\boldsymbol{X}\cdot\boldsymbol{C}\,d\boldsymbol{X})^{1/2}$$

$$\cos\theta = \frac{\boldsymbol{N}_x\cdot\boldsymbol{C}\boldsymbol{N}_y}{(\boldsymbol{N}_x\cdot\boldsymbol{C}\boldsymbol{N}_x)^{1/2}(\boldsymbol{N}_y\cdot\boldsymbol{C}\boldsymbol{N}_y)^{1/2}}$$

The difference $\Theta - \theta$ is attributed to shear.

Description of the angle between two linear elements in two configurations

For two linear elements $d\boldsymbol{X}$ and $d\boldsymbol{Y}$ in the reference configuration that intersect with angle Θ we have:

$$\cos\Theta = \frac{d\boldsymbol{X}\cdot d\boldsymbol{Y}}{\|d\boldsymbol{X}\|\,\|d\boldsymbol{Y}\|}$$

After the motion these two elements become $d\boldsymbol{x}$ and $d\boldsymbol{y}$ that intersect with angle θ :

$$\Rightarrow \quad \cos\theta = \frac{d\boldsymbol{x}\cdot d\boldsymbol{y}}{\|d\boldsymbol{x}\|\,\|d\boldsymbol{y}\|}$$

Using $d\boldsymbol{x} = \boldsymbol{F}\,d\boldsymbol{X}$; $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$; $\boldsymbol{C} = \boldsymbol{U}^2$

$$\Downarrow$$

$$\cos\theta = \frac{\boldsymbol{F}\,d\boldsymbol{X}\cdot\boldsymbol{F}\,d\boldsymbol{Y}}{\|\boldsymbol{F}\,d\boldsymbol{X}\|\,\|\boldsymbol{F}\,d\boldsymbol{Y}\|} = \frac{d\boldsymbol{X}\cdot\boldsymbol{F}^T\boldsymbol{F}\,d\boldsymbol{Y}}{\|\boldsymbol{F}d\boldsymbol{X}\|\,\|\boldsymbol{F}\,d\boldsymbol{Y}\|}$$

$$= \frac{d\boldsymbol{X}\cdot\boldsymbol{C}\,d\boldsymbol{Y}}{\|\boldsymbol{F}\,d\boldsymbol{X}\|\,\|\boldsymbol{F}\,d\boldsymbol{Y}\|}\,.$$

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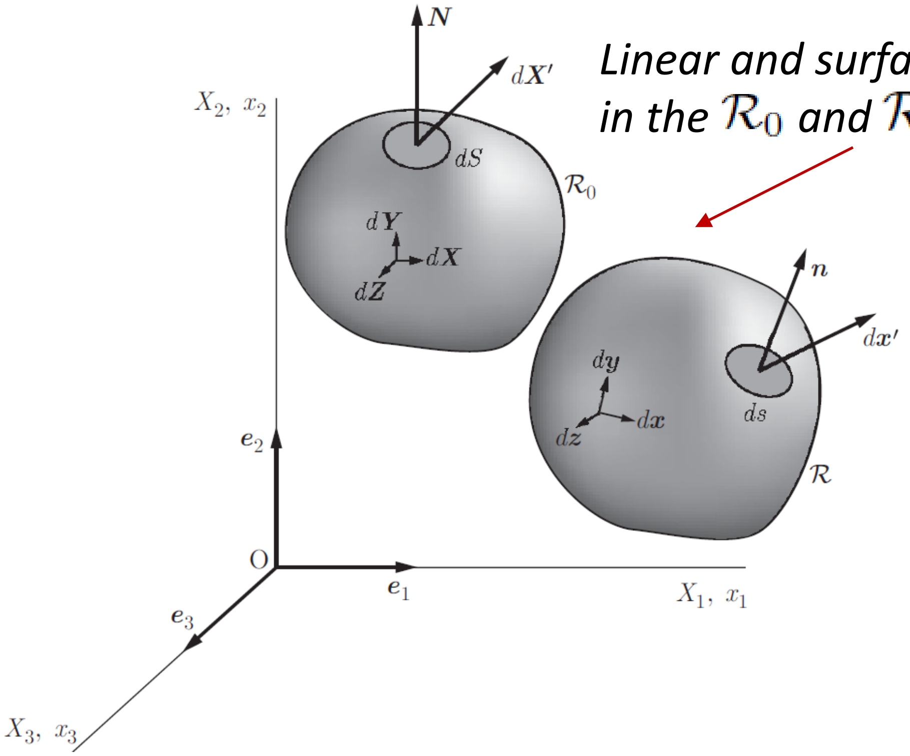

Linear and surface elements in the $\mathcal{R}_0$ *and* $\mathcal{R}$ *of the* $\mathcal{B}$

Description of volume element between two configurations

Consider three non-coplanar linear elements: $d\boldsymbol{X}$, $d\boldsymbol{Y}$, and $d\boldsymbol{Z}$. We have:

$$dV = d\boldsymbol{X} \cdot (d\boldsymbol{Y} \times d\boldsymbol{Z}) > 0$$

In the deformed configuration, the three linear elements become $d\boldsymbol{x}$, $d\boldsymbol{y}$ and $d\boldsymbol{z}$ and the volume is:

$$dv = d\boldsymbol{x} \cdot (d\boldsymbol{y} \times d\boldsymbol{z})$$

We know that $dx_i = F_{ij}\, dX_j$.

$$dv = \det \begin{pmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{pmatrix} =$$

$$= \det \begin{pmatrix} F_{1j}\, dX_j & F_{1j}\, dY_j & F_{1j}\, dZ_j \\ F_{2j}\, dX_j & F_{2j}\, dY_j & F_{2j}\, dZj \\ F_{3j}\, dX_j & F_{3j}\, dY_j & F_{3j}\, dZ_j \end{pmatrix}$$

$$\boxed{dv = \det \boldsymbol{F}\, dV = J\, dV}$$

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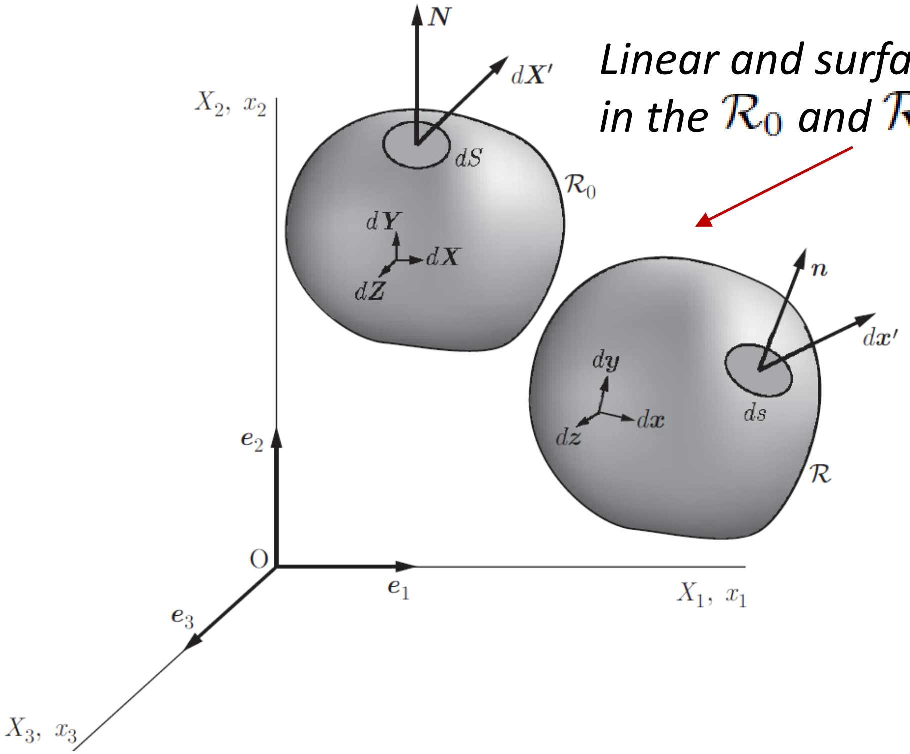


Linear and surface elements in the $\mathcal{R}_0$ and $\mathcal{R}$ of the $\mathcal{B}$

Description of surface element between two configurations (Nanson's formula)

To express the change in a surface element we start with the volume element in the refererence and deformed configurations:

$$dV = d\boldsymbol{X}' \cdot \boldsymbol{N}\, dS \qquad dv = d\boldsymbol{x}' \cdot \boldsymbol{n}\, ds$$

using $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$

Relation $dv = \det \boldsymbol{F}\, dV = J\, dV$ becomes

$$dv = \boldsymbol{F}\, d\boldsymbol{X}' \cdot \boldsymbol{n}\, ds = J\, d\boldsymbol{X}' \cdot \boldsymbol{N}\, dS$$

or

$$\left(\boldsymbol{F}^T \boldsymbol{n}\, ds - J\, \boldsymbol{N}\, dS\right) \cdot d\boldsymbol{X}' = 0$$

which is valid for any arbitrary vector $d\boldsymbol{X}'$

$$\boldsymbol{n}\, ds = J \boldsymbol{F}^{-T} \boldsymbol{N}\, dS$$

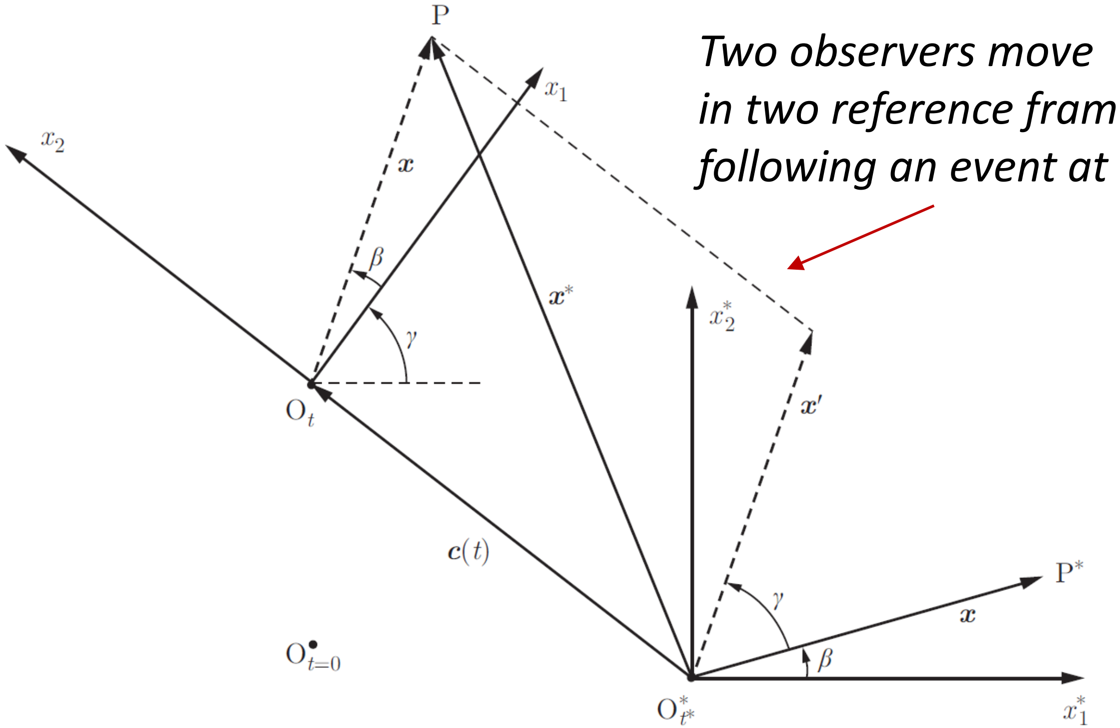
or

$$d\boldsymbol{s} = J \boldsymbol{F}^{-T} \boldsymbol{N}\, dS$$

known as Nanson's formula

Solid Mechanics: Non-linear elasticity

Objectivity of kinematic parameters



Two observers move in two reference frames following an event at P.

The same observation at P (experiment) seen by two observers in the corresponding reference frames at the same time. For the observer at R the vector position is x. For the observer at R^* we must take into account the rotation of R with respect to R^*.

Consider an event viewed by two observers R and R^*, and noted respectively by $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}^*, t^*)$.

The motion between two observers is a function of space and time (effects due to relativity are negligible).

The two observers measure the same distance between two events as well as the same time intervals between events.

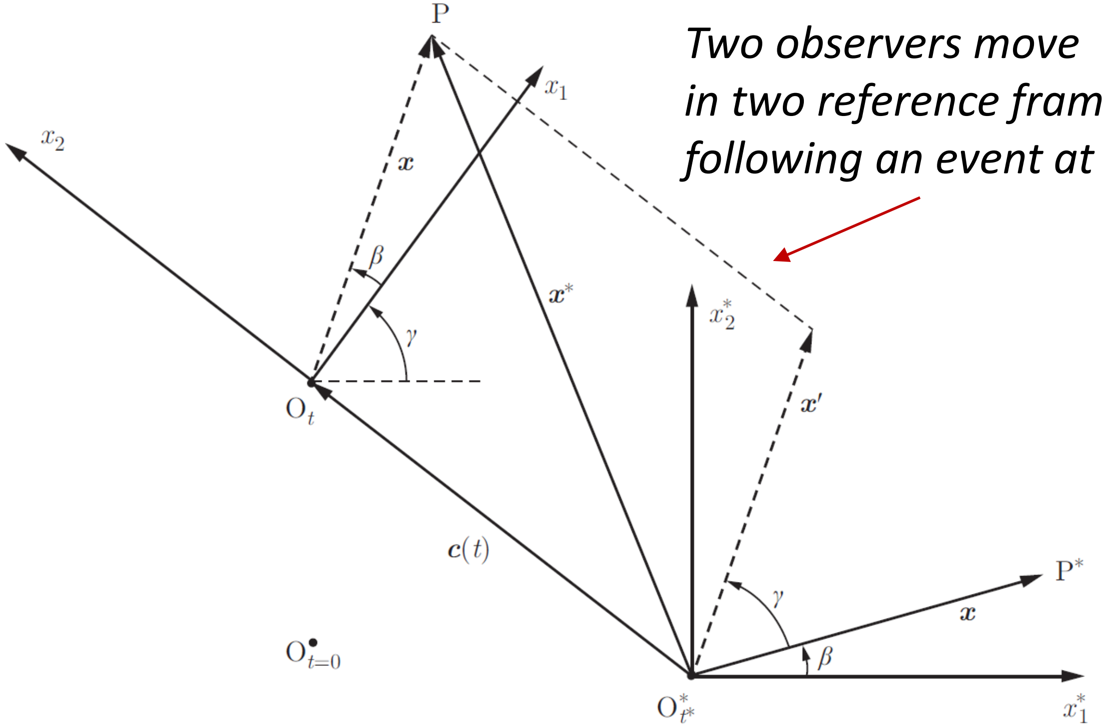
The most general transformation between the two observations of the same event is given by:

$$\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}(t) \quad \text{where} \quad t^* = t - \alpha$$

Here $\boldsymbol{Q}(t)$ is an orthogonal tensor with time as a parameter, $\boldsymbol{c}(t)$ is a vector and α is a scalar.

Solid Mechanics: Non-linear elasticity

Objectivity of kinematic parameters



Two observers move in two reference frames following an event at P.

The motion of the body $\mathcal{B}$ described by $\chi(\boldsymbol{X}, t)$ according to the first observer, is described by the second observer as $\chi^*(\boldsymbol{X}, t^*)$.

The two descriptions are related as follows:

$$\chi^*(\boldsymbol{X}, t^*) = \boldsymbol{Q}(t)\chi(\boldsymbol{X}, t) + \boldsymbol{c}(t)$$

To examine the ramifications of this relation we consider two events reordered by:

$$\mathrm{R} \ : (\boldsymbol{x}_1, t) \ , (\boldsymbol{x}_2, t) \ ; \ \mathrm{R}^*: (\boldsymbol{x}_1^*, t) \ \ (\boldsymbol{x}_2^*, t)$$

The relative positions of these events are:

$$\mathrm{R} : \boldsymbol{u} = \boldsymbol{x}_2 - \boldsymbol{x}_1 \ ; \ \mathrm{R}^*: \boldsymbol{u}^* = \boldsymbol{x}_2^* - \boldsymbol{x}_1^*$$

$$\boxed{\boldsymbol{u}^* = \boldsymbol{Q}\boldsymbol{u}}$$

$$\boxed{\begin{aligned}\boldsymbol{u}^* \cdot \boldsymbol{u}^* &= (\boldsymbol{Q}\boldsymbol{u}) \cdot (\boldsymbol{Q}\boldsymbol{u}) \\ &= \boldsymbol{u} \cdot (\boldsymbol{Q}^T\boldsymbol{Q})\boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{u}\end{aligned}}$$

$$\|\boldsymbol{u}^*\| = \|\boldsymbol{u}\|$$

The transformation $\boldsymbol{u}^* = \boldsymbol{Q}\boldsymbol{u}$ is that of a rigid body.

Solid Mechanics: Non-linear elasticity

Objective fields

A vector field transformed according to:

$u^* = Qu$ is called **spatially objective vector field**.

Using this definition we can define a spatially objective 2nd tensor filed. v

Two spatially objective vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ seen by the observer R , are related by:

$$w = Lv.$$

Since they are objective, the observer R^* sees

$$w^* = Qw \quad \text{and} \quad w^* = L^* v^*$$

$$w^* = Qw = QLv = QLQ^T v^*$$

$$v^* = Qv$$

$$L^* = QLQ^T$$

A tensor transformed according to the last relation is **spatially objective tensor** or independent of the reference frame.

In summary

A scalar quantity ϕ is objective if and only if (iif) $\phi^* = \phi;$

A vector quantity f is **materially objective** iff $f^* = f;$

A vector quantity f is **spatially objective** iff $f^* = Qf;$

A tensor quantity T is **materially objective** iff $T^* = T;$

A tensor quantity T is **spatially objective** iff $T^* = QTQ^T$

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Objectivity of velocity and acceleration

We have for the velocity $\boldsymbol{V}(\boldsymbol{X},t) = \dot{\chi}(\boldsymbol{X},t)$

and acceleration $\boldsymbol{A}(\boldsymbol{X},t) = \ddot{\chi}(\boldsymbol{X},t)$

$$\chi^*(\boldsymbol{X},t^*) = \boldsymbol{Q}(t)\chi(\boldsymbol{X},t) + \boldsymbol{c}(t)$$

$$\begin{aligned} \boldsymbol{V}^*(\boldsymbol{X},t^*) &= \boldsymbol{Q}(t)\boldsymbol{V}(\boldsymbol{X},t) + \dot{\boldsymbol{c}}(t) + \dot{\boldsymbol{Q}}(t)\chi(\boldsymbol{X},t) \\ \boldsymbol{A}^*(\boldsymbol{X},t^*) &= \ddot{\chi}^*(\boldsymbol{X},t^*) \\ &= \boldsymbol{Q}(t)\ddot{\chi}(\boldsymbol{X},t) + \ddot{\boldsymbol{c}}(t) + \ddot{\boldsymbol{Q}}(t)\chi(\boldsymbol{X},t) \\ &\quad + 2\dot{\boldsymbol{Q}}(t)\boldsymbol{V}(X,t)\,. \end{aligned}$$

The definitions of the velocity and acceleration are relative and inextricably linked to the observer.

For the deformation gradient tensor we have

$$\begin{aligned} \underline{\boldsymbol{F}^*(\boldsymbol{X},t^*)} &= \frac{\partial\chi^*(\boldsymbol{X},t)}{\partial\boldsymbol{X}} \\ &= \frac{\partial\chi^*(\boldsymbol{X},t)}{\partial\chi(\boldsymbol{X},t)}\,\frac{\partial\chi(\boldsymbol{X},t)}{\partial\boldsymbol{X}} \\ &= \underline{\boldsymbol{Q}(t)\boldsymbol{F}(\boldsymbol{X},t)} \end{aligned}$$

and

$$J^* = \det\boldsymbol{F}^*(\boldsymbol{X},t^*) = \det\boldsymbol{F}(\boldsymbol{X},t) = J$$

Starting from the definitions of the corresponding Tensors it can be shown that:

$$\boldsymbol{C}^* = \boldsymbol{C} \qquad \boldsymbol{E}^* = \boldsymbol{E}$$

$$\boldsymbol{c}^* = \boldsymbol{Q}\boldsymbol{c}\boldsymbol{Q}^T \qquad \boldsymbol{e}^* = \boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T$$

Solid Mechanics: Non-linear elasticity

Piola-Kirchhoff stress tensors

The Cauchy stress tensor $\boldsymbol{\sigma}$, is expressed with respect to the current configuration $\mathcal{R}$ i.,e. it is the real stress.

The principles of momentum and angular momentum are formulated with respect to the current configurtion.

Problems in solid mechanics require a formulation with respect to the initial configurtion $\mathcal{R}_0$.

This is because: (a) it is difficult to know the deformed condition of a solid beforehand, (b) it is more convenient to analyze the experimental response of a solid with respect to its undeformed configuration.

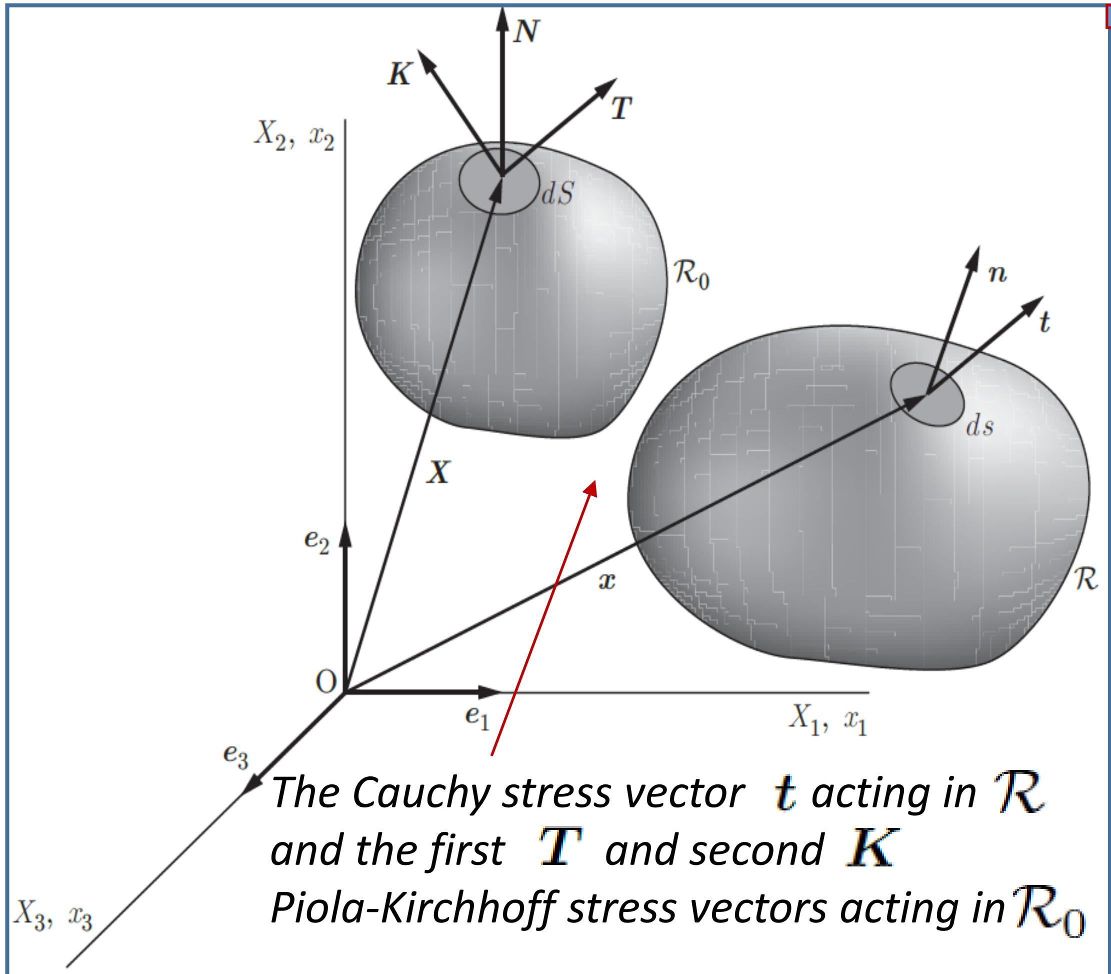
There is not simply a change of variables in the equations of motion and the Cauchy stress components using:

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t)$$

Measurements of stresses in the undeformed configuration have been proposed for the study of problems in solid mechanics.

These are the first and second Piola-Kirchhoff stress tensors.

Solid Mechanics: Non-linear elasticity



The Cauchy stress vector $\boldsymbol{t}$ *acting in* $\mathcal{R}$ *and the first* $\boldsymbol{T}$ *and second* $\boldsymbol{K}$ *Piola-Kirchhoff stress vectors acting in* $\mathcal{R}_0$

Here $\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})$ is the Cauchy stress vector acting on the actual surface element $\boldsymbol{n}\,ds$ at $\boldsymbol{x}$.

To this vector we associate the vector $\boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N})$ called the first Piola-Kirchhoff stress vector, to the corresponding reference surface element $\boldsymbol{N}\,dS$, and related to $\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})$ as follows:

$$\boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N}(\boldsymbol{X}))\,dS = \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n}(\boldsymbol{x},t))\,ds$$

dS and ds are positive. Thus, $\boldsymbol{T}$ and $\boldsymbol{t}$ have the same direction but $\|\boldsymbol{T}\|$ and $\|\boldsymbol{t}\|$ are not generally the same.

Note that the stress vector is not real (often called pseudo-stress).

Using Cauchy's relation $t_i(\boldsymbol{x},t,\boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x},t)n_j$

and Nanson's formula $\boldsymbol{n}\,ds = J\boldsymbol{F}^{-T}\boldsymbol{N}\,dS$

$$\begin{aligned}\boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N})\,dS &= \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\,ds = \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}\,ds \\ &= J(\boldsymbol{X},t)\boldsymbol{\sigma}(\chi(\boldsymbol{X},t),t)\boldsymbol{F}^{-T}\boldsymbol{N}\,dS\end{aligned}$$

Solid Mechanics: Non-linear elasticity

Piola-Kirchhoff stress tensors

$$\boxed{\begin{aligned} \boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N})\,dS &= \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\,ds = \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}\,ds \\ &= J(\boldsymbol{X},t)\boldsymbol{\sigma}(\chi(\boldsymbol{X},t),t)\boldsymbol{F}^{-T}\boldsymbol{N}\,dS \end{aligned}} \Longrightarrow \boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N}) = \boldsymbol{P}(\boldsymbol{X},t)\boldsymbol{N}$$

The First Piola-Kirchhoff stress tensor defined as:

$$\boldsymbol{P}(\boldsymbol{X},t) = J(\boldsymbol{X},t)\boldsymbol{\sigma}(\chi(\boldsymbol{X},t),t)\boldsymbol{F}^{-T}$$

Using the symmetry of the Cauchy's stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ it is shown below that: $\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T$

$$\boxed{\begin{aligned} &\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} \Rightarrow \boldsymbol{P}\,\boldsymbol{F}^T = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}\boldsymbol{F}^T = J\boldsymbol{\sigma} \\ &J\boldsymbol{\sigma} = J\boldsymbol{\sigma}^T \\ &\Rightarrow \boldsymbol{P}\boldsymbol{F}^T = \left(\boldsymbol{P}\boldsymbol{F}^T\right)^T \Rightarrow \boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T \end{aligned}}$$

Solid Mechanics: Non-linear elasticity

Objectivity of the tensor $\boldsymbol{\sigma}$

We consider Cauchy's relation $t_i(\boldsymbol{x},t,\boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x},t)n_j$ as seen by two observers R and R^*.

We assume that vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ are objective and are transformed as:

$$\boldsymbol{t}^* = \boldsymbol{Qt} \quad ; \quad \boldsymbol{n}^* = \boldsymbol{Qn}$$

From $\boldsymbol{t}^* = \boldsymbol{\sigma}^*\boldsymbol{n}^*$ we have $\boldsymbol{Qt} = \boldsymbol{\sigma}^*\boldsymbol{Qn}$

We multiply $\boldsymbol{t} = \boldsymbol{\sigma n}$ by $\boldsymbol{Q}$ to obtain: $\boldsymbol{Qt} = \boldsymbol{Q\sigma n}$

Comparing the two last results we have:

$$\boldsymbol{\sigma}^* = \boldsymbol{Q\sigma Q}^T$$

⇒ the Cauchy's stress tensor is objective.

Objectivity of the tensor $\boldsymbol{P}$

To check the objectivity of $\boldsymbol{P}$ we start with

$$\boldsymbol{P}^*\boldsymbol{F}^{*T} = J^*\boldsymbol{\sigma}^*$$

and use

$$\boldsymbol{F}^* = \boldsymbol{QF} \quad ; \quad J^* = J \quad ; \quad \boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$

$$\boldsymbol{P}^*(\boldsymbol{QF})^T = J\boldsymbol{Q\sigma Q}^T$$

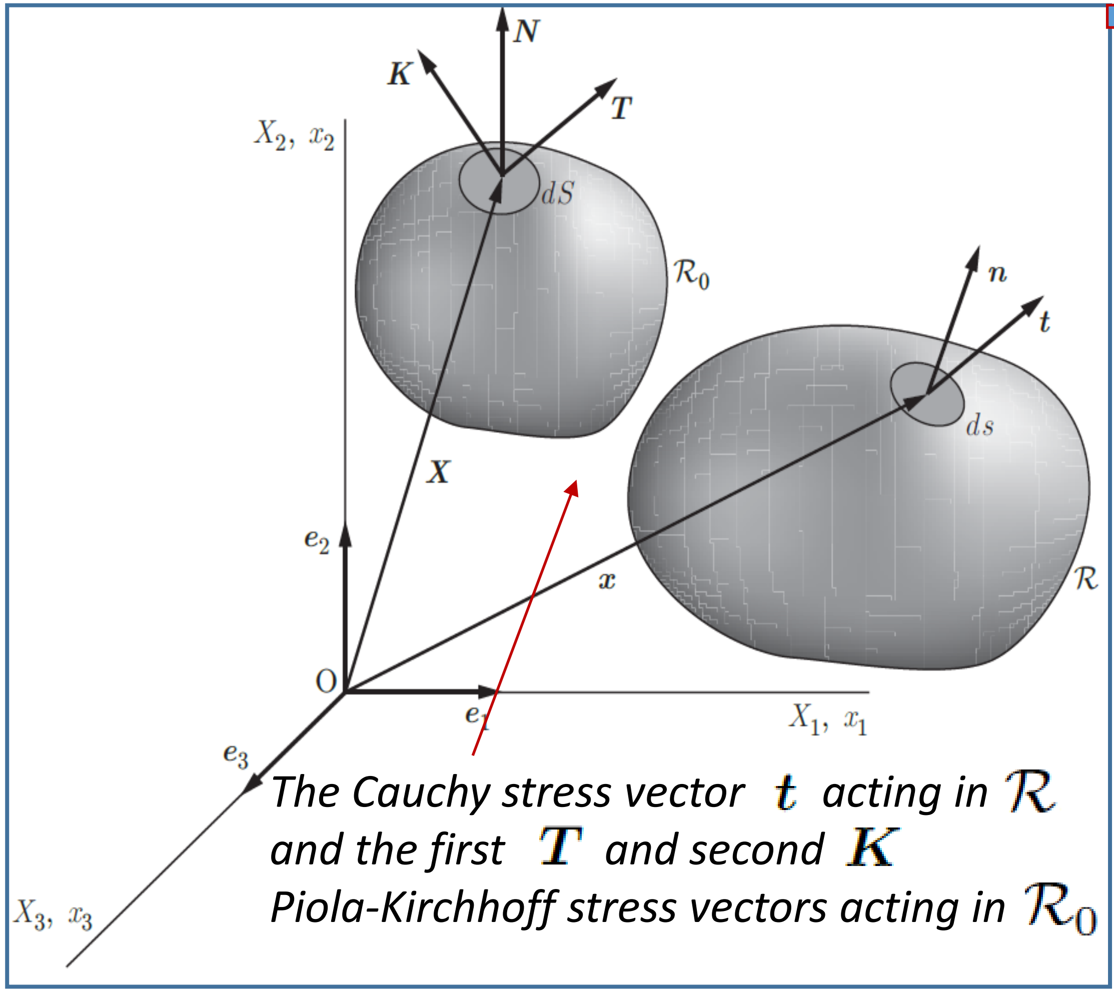
$$\boldsymbol{P}^*\boldsymbol{F}^T\boldsymbol{Q}^T = \boldsymbol{Q}J\boldsymbol{\sigma}\boldsymbol{Q}^T = \boldsymbol{QPF}^T\boldsymbol{Q}^T$$

$$\boldsymbol{P}^* = \boldsymbol{QP}\,.$$

⇓

The first Piola-Kirchhoff stress tensor is not objective.

Solid Mechanics: Non-linear elasticity



The Cauchy stress vector $\boldsymbol{t}$ *acting in* $\mathcal{R}$ *and the first* $\boldsymbol{T}$ *and second* $\boldsymbol{K}$ *Piola-Kirchhoff stress vectors acting in* $\mathcal{R}_0$

Here $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ is the Cauchy stress vector acting on the actual surface element $\boldsymbol{n}\, ds$ at $\boldsymbol{x}$.

To this vector we associate the vector called the second Piola-Kirchhoff stress vector, to the corresponding reference surface element $\boldsymbol{N}\, dS$, and related to $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ as follows:

$$\boldsymbol{K}(\boldsymbol{X}, t, \boldsymbol{N})\, dS = \boldsymbol{F}^{-1}(\boldsymbol{X}, t)\, \boldsymbol{t}\big(\chi(\boldsymbol{X}, t),\ t,\ \boldsymbol{n}(\boldsymbol{X}, t)\big)\, ds$$

Here $\boldsymbol{K}$ expresses the contact force per unit reference surface transformed by $\boldsymbol{F}^{-1}$: Expressing

$$\boldsymbol{K}(\boldsymbol{X}, t, \boldsymbol{N}) = \boldsymbol{S}(\boldsymbol{X}, t)\boldsymbol{N}$$

We can define the second Piola-Kirchhoff tensor $\boldsymbol{S}$ given below:

Using Cauchy's relation: $t_i(\boldsymbol{x}, t, \boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x}, t) n_j$

and Nanson's formula: $\boldsymbol{n}\, ds = J \boldsymbol{F}^{-T} \boldsymbol{N}\, dS$

$$\boldsymbol{S}(\boldsymbol{X}, t) = J(\boldsymbol{X}, t)\boldsymbol{F}^{-1}(\boldsymbol{X}, t)\boldsymbol{\sigma}\big(\chi(\boldsymbol{X}, t), t\big)\boldsymbol{F}^{-T}(\boldsymbol{X}, t)$$
$$= \boldsymbol{F}^{-1}(\boldsymbol{X}, t)\boldsymbol{P}(\boldsymbol{X}, t)\,.$$
which is symmetric

Solid Mechanics: Non-linear elasticity

Linearization of the stress tensors

It is important to check what are the effects of the Kinematic linearization on the three stress tensors.

From the relation:

$$\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T$$

We express the tensor $\boldsymbol{P}$ in index form as:

$$P_{mk} = F_{mi}(P_{ij})^T(F_{jk})^{-T} = F_{mi}P_{ji}F^{-1}_{kj}$$

Using: $F_{ij} = \delta_{ij} + \frac{\partial U_i}{\partial X_j}$; $F^{-1}_{ij} = \delta_{ij} - \frac{\partial u_i}{\partial x_j}$

and $\frac{\partial U_i}{\partial X_j} = \frac{\partial u_i}{\partial x_j} + O(\varepsilon^2) \simeq \frac{\partial u_i}{\partial x_j}$

$$P_{mk} = P_{km} - P_{jm}\frac{\partial U_k}{\partial X_j} + P_{ki}\frac{\partial U_m}{\partial X_i} - P_{ji}\frac{\partial U_m}{\partial X_i}\frac{\partial U_k}{\partial X_j}$$

Similarly the second Piolla-Kirchhoff stress tensor:

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P}$$

takes the form in index notation:

$$S_{ij} = F^{-1}_{ik}P_{kj} = \left(\delta_{ik} - \frac{\partial U_i}{\partial X_k}\right)P_{kj} = P_{ij} - P_{kj}\frac{\partial U_i}{\partial X_k}$$

For the Cauchy stress tensor, we express it using:

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$

as $\sigma_{ij} = J^{-1}P_{ik}(F_{kj})^T = J^{-1}P_{ik}F_{jk}$

Solid Mechanics: Non-linear elasticity

Linearization of the stress tensors

$$\sigma_{ij} = J^{-1} P_{ik} (F_{kj})^T = J^{-1} P_{ik} F_{jk}$$

Using

$$F_{ij} = \delta_{ij} + \frac{\partial U_i}{\partial X_j} \;; \quad F_{ij}^{-1} = \delta_{ij} - \frac{\partial u_i}{\partial x_j}$$

$$F_{ij} = \delta_{ij} + O(\varepsilon) \;; \quad F_{ij}^{-1} = \delta_{ij} - O(\varepsilon)$$

$$J \approx 1 + O(\varepsilon)$$

$$\Downarrow$$

$$\sigma_{ij} = J^{-1} P_{ik} \left(\delta_{jk} + \frac{\partial U_j}{\partial X_k} \right)$$

$$= J^{-1} (P_{ij} + P_{ik} \frac{\partial U_j}{\partial X_k}) \approx P_{ij} + P_{ik} \frac{\partial U_j}{\partial X_k}$$

If we neglect the terms with the displacement gradient on the expressions:

$$P_{mk} = P_{km} - P_{jm} \frac{\partial U_k}{\partial X_j} + P_{ki} \frac{\partial U_m}{\partial X_i} - P_{ji} \frac{\partial U_m}{\partial X_i} \frac{\partial U_k}{\partial X_j}$$

$$S_{ij} = F_{ik}^{-1} P_{kj} = \left(\delta_{ik} - \frac{\partial U_i}{\partial X_k} \right) P_{kj} = P_{ij} - P_{kj} \frac{\partial U_i}{\partial X_k}$$

$$\sigma_{ij} = J^{-1} P_{ik} \left(\delta_{jk} + \frac{\partial U_j}{\partial X_k} \right)$$

$$= J^{-1} (P_{ij} + P_{ik} \frac{\partial U_j}{\partial X_k}) \approx P_{ij} + P_{ik} \frac{\partial U_j}{\partial X_k}$$

$$\Downarrow$$

$$P_{mk} \approx P_{km} \qquad S_{ij} \approx P_{ij} \qquad \sigma_{ij} \approx P_{ij}$$

Solid Mechanics: Non-linear elasticity

Two approaches to constitutive equations for isotropic elastic material:

1. **Cauchy elasticity, or Cauchy elastic material**:

   It is based on purely theoretical considerations and without any reference to thermodynamics.

   We arrive at the general form $\boldsymbol{\sigma} = K\,(e)$ ($stress\ \boldsymbol{\sigma}\ \ and\ \ strain\ \ \boldsymbol{e}$) and use the *Rivlin - Ericksen representation theorem*:

$$\boldsymbol{\sigma} = k_0\big(\mathrm{I}_1(\boldsymbol{e}),\, \mathrm{I}_2(\boldsymbol{e}),\, \mathrm{I}_3(\boldsymbol{e})\big)\boldsymbol{I} + k_1\big(\mathrm{I}_1(\boldsymbol{e}),\, \mathrm{I}_2(\boldsymbol{e}),\, \mathrm{I}_3(\boldsymbol{e})\big)\boldsymbol{e} + k_2\big(\mathrm{I}_1(\boldsymbol{e}),\, \mathrm{I}_2(\boldsymbol{e}),\, \mathrm{I}_3(\boldsymbol{e})\big)\boldsymbol{e}^2$$

where

$$k_p = k_p\big(\mathrm{I}_1(\boldsymbol{e}),\, \mathrm{I}_2(\boldsymbol{e}),\, \mathrm{I}_3(\boldsymbol{e})\big) \qquad p = 0, 1, 2$$

are scalars of the invariants of the Euler-Almansi strain tensor

Solid Mechanics: Non-linear elasticity

Two approaches to constitutive equations for isotropic elastic material:

2. **finite hyperelasticity, or hyperelasticity or Green elastic material**:

It is based on the hypothesis of the **existence of an energy function**.

From thermodynamic considerations, and this function we define the constitutive equation as:

First Piola-Kirchhof Stress Tensor

Energy function

$$\boldsymbol{P} = \frac{\partial \mathcal{W}(\boldsymbol{F})}{\partial \boldsymbol{F}} \, .$$

Deformation gradient tensor

Recalling that $\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$

$$\Longrightarrow \boxed{\boldsymbol{\sigma} = J^{-1}\frac{\partial \mathcal{W}(\boldsymbol{F})}{\partial \boldsymbol{F}}\,\boldsymbol{F}^{T} \, .}$$

Solid Mechanics: Non-linear elasticity

Constraints on the energy function:

It should be independent of the reference frame.

→ For two observers $\mathcal{W}(\boldsymbol{F}) = \mathcal{W}(\boldsymbol{F}^{\star}) = \mathcal{W}(\boldsymbol{Q}\boldsymbol{F})\,.$

We replace $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$ and consider $\boldsymbol{Q} = \boldsymbol{R}^{T}$

↓

$$\boxed{\mathcal{W}(\boldsymbol{F}) = \mathcal{W}(\boldsymbol{R}^{T}\boldsymbol{R}\boldsymbol{U}) = \mathcal{W}(\boldsymbol{U})}$$

This is the necessary and sufficient condition for the objectivity of the strain energy function

$$\xrightarrow{\boldsymbol{U} = \boldsymbol{C}^{1/2}} \mathcal{W}(\boldsymbol{F}) = \mathcal{W}(\boldsymbol{U}) = \widehat{\mathcal{W}}(\boldsymbol{C})\,.$$

Solid Mechanics: Non-linear elasticity

We formulate constitutive equations using the metric tensor $\boldsymbol{C}$ and not $\boldsymbol{F}$

→ It is necessary to express $\partial \mathcal{W}(\boldsymbol{F})/\partial \boldsymbol{F}$ as a function of $\boldsymbol{C}$

$$\mathcal{W}(\boldsymbol{F}) = \mathcal{W}(\boldsymbol{U}) = \widehat{\mathcal{W}}(\boldsymbol{C})\,. \longrightarrow \frac{\partial \mathcal{W}(\boldsymbol{F})}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} = \frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}} : \dot{\boldsymbol{C}}\,.$$

Taking into consideration $\boldsymbol{C} = \boldsymbol{C}^T = \boldsymbol{F}^T \boldsymbol{F}$ it can be shown that

$$\left(\frac{\partial \mathcal{W}(\boldsymbol{F})}{\partial \boldsymbol{F}}\right)^T = 2\frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}} \boldsymbol{F}^T\,.$$

Since both $\boldsymbol{C},\ \partial \widehat{\mathcal{W}}(\boldsymbol{C})/\partial \boldsymbol{C}$ are symmetric

→

$$\boxed{\frac{\partial \mathcal{W}(\boldsymbol{F})}{\partial \boldsymbol{F}} = 2\boldsymbol{F}\frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}}}$$

Solid Mechanics: Non-linear elasticity

Using $\boxed{\dfrac{\partial \mathcal{W}(\boldsymbol{F})}{\partial \boldsymbol{F}} = 2\boldsymbol{F}\dfrac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}}}$

the original expression $\boldsymbol{P} = \dfrac{\partial \mathcal{W}(\boldsymbol{F})}{\partial \boldsymbol{F}}$ becomes

$$\boldsymbol{P} = 2\boldsymbol{F}\frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}} .$$

We can also use the second Piola-Kirchhoff tensor from $\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P}$

$$\boldsymbol{S} = 2\frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}}$$

This is more convenient because it does not contain $\boldsymbol{P}$ and $\boldsymbol{F}$ tensors since they are not symmetric

Solid Mechanics: Non-linear elasticity

Isotropic Hyperelastic Materials

In the case of isotropic medium, the strain energy function should satisfy the condition,

$$\widehat{W}(\boldsymbol{C}) = \widehat{W}(\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T)\,.$$

This equality implies that $\widehat{W}(\boldsymbol{C})$ is an isotropic function of the symmetric tensor $\boldsymbol{C}$.

Because of the isotropy, the energy function can be written in terms of the principal invariants of $\boldsymbol{C}$ ($\lambda_1^2,\ \lambda_2^2,\ \lambda_3^2$ are the principal values).

invariants of $\boldsymbol{C}$ →

$$I_1(\boldsymbol{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2(\boldsymbol{C}) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3(\boldsymbol{C}) = \lambda_1^2\lambda_2^2\lambda_3^2\,.$$

→

$$\boxed{\widehat{W}(\boldsymbol{C}) = \Phi\big(I_1(\boldsymbol{C}),\ I_2(\boldsymbol{C}),\ I_3(\boldsymbol{C})\big)}$$

Solid Mechanics: Non-linear elasticity

Isotropic Hyperelastic Materials

With $\boldsymbol{S} = 2\dfrac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}}$

and $\widehat{\mathcal{W}}(\boldsymbol{C}) = \Phi\big(I_1(\boldsymbol{C}),\, I_2(\boldsymbol{C}),\, I_3(\boldsymbol{C})\big)$

we need

$$\frac{\partial \Phi(\boldsymbol{C})}{\partial \boldsymbol{C}} = \frac{\partial \Phi}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{C}} + \frac{\partial \Phi}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{C}} + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \boldsymbol{C}}$$

with $\dfrac{\partial I_1}{\partial \boldsymbol{C}} = \boldsymbol{I}\,,\qquad \dfrac{\partial I_2}{\partial \boldsymbol{C}} = I_1\boldsymbol{I} - \boldsymbol{C}\,,\qquad \dfrac{\partial I_3}{\partial \boldsymbol{C}} = I_3\boldsymbol{C}^{-1}\,.$

$$\frac{\partial \Phi(\boldsymbol{C})}{\partial \boldsymbol{C}} = I_3\frac{\partial \Phi}{\partial I_3}\,\boldsymbol{C}^{-1} + \left(\frac{\partial \Phi}{\partial I_1} + I_1\frac{\partial \Phi}{\partial I_2}\right)\boldsymbol{I} - \frac{\partial \Phi}{\partial I_2}\,\boldsymbol{C}\,.$$

Solid Mechanics: Non-linear elasticity

Isotropic Hyperelastic Materials

With
$$\boldsymbol{S} = 2\frac{\partial \widehat{W}(\boldsymbol{C})}{\partial \boldsymbol{C}} = 2\frac{\partial \Phi(\boldsymbol{C})}{\partial \boldsymbol{C}}$$

$$\frac{\partial \Phi(\boldsymbol{C})}{\partial \boldsymbol{C}} = I_3 \frac{\partial \Phi}{\partial I_3} \boldsymbol{C}^{-1} + \left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2} \right) \boldsymbol{I} - \frac{\partial \Phi}{\partial I_2} \boldsymbol{C} \, .$$

$$\boldsymbol{S} = 2 \left(I_3 \frac{\partial \Phi}{\partial I_3} \boldsymbol{C}^{-1} + \left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2} \right) \boldsymbol{I} - \frac{\partial \Phi}{\partial I_2} \boldsymbol{C} \right)$$

At the reference configuration

$$\boldsymbol{S} = 0 \quad \Rightarrow \quad \boldsymbol{C} = \boldsymbol{I}$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 1$$

$$I_1 = 3, \quad I_2 = 3, \quad I_3 = 1$$

$$\frac{\partial \Phi}{\partial I_1} + 2\frac{\partial \Phi}{\partial I_2} + \frac{\partial \Phi}{\partial I_3} = 0 \, .$$

Solid Mechanics: Non-linear elasticity

Isotropic Hyperelastic Materials

Consequently, when the energy function of a certain material is known, the constitutive response is established by one of the two.

$$\boldsymbol{S} = 2\left(I_3 \frac{\partial \Phi}{\partial I_3} \boldsymbol{C}^{-1} + \left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2} \right) \boldsymbol{I} - \frac{\partial \Phi}{\partial I_2} \boldsymbol{C} \right)$$

Introduce ***S*** in

$$\boldsymbol{\sigma} = \boldsymbol{J}^{-1} \boldsymbol{F} \boldsymbol{S} \boldsymbol{F}^{T}$$

$$\boldsymbol{\sigma} = 2J^{-1}\left(I_3(\boldsymbol{c}) \frac{\partial \Phi}{\partial I_3(\boldsymbol{c})} \boldsymbol{I} + \left(\frac{\partial \Phi}{\partial I_1(\boldsymbol{c})} + I_1(\boldsymbol{c}) \frac{\partial \Phi}{\partial I_2(\boldsymbol{c})} \right) \boldsymbol{c} - \frac{\partial \Phi}{\partial I_2(\boldsymbol{c})} \boldsymbol{c}^2 \right)$$

$\boldsymbol{c}$ left Cauchy-Green deformation tensor

$\boldsymbol{C}$ right Cauchy-Green deformation tensor

They have the same principal stretches $\lambda_1^2,\ \lambda_2^2,\ \lambda_3^2$

Thus the corresponding invariants are the same.

Solid Mechanics: Non-linear elasticity

Isotropic Hyperelastic Materials

For an isotropic material, the strain energy function can be expressed also in terms of the principal stretches:

$$\widehat{\mathcal{W}}(\boldsymbol{C}) = \phi(\lambda_1, \lambda_2, \lambda_3)$$

We start again from →
$$\boldsymbol{S} = 2\frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}}$$

and differentiate →
$$\frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}} = \frac{\partial \phi}{\partial \lambda_i^2}\frac{\partial \lambda_i^2}{\partial \boldsymbol{C}} = \frac{1}{2\lambda_i}\frac{\partial \phi}{\partial \lambda_i}\frac{\partial \lambda_i^2}{\partial \boldsymbol{C}}$$

With →
$$\frac{\partial \lambda_i^2}{\partial \boldsymbol{C}} = \boldsymbol{A}_i \otimes \boldsymbol{A}_i$$

$$\boxed{\frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}} = \sum_{i=1}^{3} \frac{1}{2\lambda_i}\frac{\partial \phi}{\partial \lambda_i}\boldsymbol{A}_i \otimes \boldsymbol{A}_i}$$

λ_i^2 principal values of $\boldsymbol{C}$,

$\boldsymbol{A}_i$ corresponding principal directions

$$\boxed{\boldsymbol{S} = \sum_{i=1}^{3} \frac{1}{\lambda_i}\frac{\partial \phi}{\partial \lambda_i}\boldsymbol{A}_i \otimes \boldsymbol{A}_i}$$

Solid Mechanics: Non-linear elasticity

Isotropic Hyperelastic Materials

$$\boxed{\boldsymbol{S} = \sum_{i=1}^{3} \frac{1}{\lambda_i} \frac{\partial \phi}{\partial \lambda_i} \boldsymbol{A}_i \otimes \boldsymbol{A}_i}$$

$$S_i = \frac{1}{\lambda_i} \frac{\partial \phi}{\partial \lambda_i}$$

In terms of the principal values

$$\begin{aligned} \boldsymbol{P} = \boldsymbol{F}\boldsymbol{S} &= \boldsymbol{F}\left(\sum_{i=1}^{3} \frac{1}{\lambda_i} \frac{\partial \phi}{\partial \lambda_i} \boldsymbol{A}_i \otimes \boldsymbol{A}_i\right) \\ &= \sum_{i=1}^{3} \frac{1}{\lambda_i} \frac{\partial \phi}{\partial \lambda_i} (\boldsymbol{F}\boldsymbol{A})_i \otimes \boldsymbol{A}_i = \sum_{i=1}^{3} \frac{\partial \phi}{\partial \lambda_i} \boldsymbol{b}_i \otimes \boldsymbol{A}_i \,. \end{aligned}$$

$$\boxed{P_i = \frac{\partial \phi}{\partial \lambda_i} \,.}$$

$$\begin{aligned} \boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{P}^T &= J^{-1}\boldsymbol{F}\left(\sum_{i=1}^{3} \frac{\partial \phi}{\partial \lambda_i} (\boldsymbol{b}_i \otimes \boldsymbol{A}_i)^T\right) \\ &= J^{-1}\left(\sum_{i=1}^{3} \frac{\partial \phi}{\partial \lambda_i} \boldsymbol{F}\boldsymbol{A}_i \otimes \boldsymbol{b}_i\right) = J^{-1}\left(\sum_{i=1}^{3} \lambda_i \frac{\partial \phi}{\partial \lambda_i} \boldsymbol{b}_i \otimes \boldsymbol{b}_i\right) \end{aligned}$$

$$\boxed{\sigma_i = J^{-1} \lambda_i \frac{\partial \phi}{\partial \lambda_i}}$$

Solid Mechanics: Non-linear elasticity

Recall: For isotropic Hyperelastic Materials:

We have the two constitutive Equations

$$\boldsymbol{S} = 2\left(I_3\frac{\partial\Phi}{\partial I_3}\boldsymbol{C}^{-1} + \left(\frac{\partial\Phi}{\partial I_1} + I_1\frac{\partial\Phi}{\partial I_2}\right)\boldsymbol{I} - \frac{\partial\Phi}{\partial I_2}\boldsymbol{C}\right)$$

in $\boldsymbol{\sigma} = \boldsymbol{J}^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T$

$$\boldsymbol{\sigma} = 2J^{-1}\left(I_3(\boldsymbol{c})\frac{\partial\Phi}{\partial I_3(\boldsymbol{c})}\boldsymbol{I} + \left(\frac{\partial\Phi}{\partial I_1(\boldsymbol{c})} + I_1(\boldsymbol{c})\frac{\partial\Phi}{\partial I_2(\boldsymbol{c})}\right)\boldsymbol{c} - \frac{\partial\Phi}{\partial I_2(\boldsymbol{c})}\boldsymbol{c}^2\right)$$

When the energy function of a certain material is known, the constitutive response is established by one of the two relations.

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

The volume remains unchanged during deformation called isochoric motion.

Examples: rubbers, certain soft biological tissues

Incompressibility condition:

$$J = \frac{dv}{dV} = \lambda_1 \lambda_2 \lambda_3 = 1$$

Or the third invariant of ***C*** or ***c***

$$I_3 = {\lambda_1}^2 {\lambda_2}^2 {\lambda_3}^2 = 1$$

The Constitutive Equations become

$$\boldsymbol{S} = -p\,\boldsymbol{C}^{-1} + 2\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\boldsymbol{I} - 2\frac{\partial \Phi}{\partial I_2}\boldsymbol{C}$$

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\boldsymbol{c} - 2\frac{\partial \Phi}{\partial I_2}\boldsymbol{c}^2$$

It is a constant, associated with pressure (does not produce work) and is calculated from equilibrium and BC.

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

It is useful to express the principal stresses as a function of the principal stretches.

Note: In isotropic materials, the principal directions of stresses and the principal stretches coincide

The equation

$$\boxed{\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\boldsymbol{c} - 2\frac{\partial \Phi}{\partial I_2}\,\boldsymbol{c}^2}$$

becomes

$$\boxed{\sigma_i = -p + 2\left(\frac{\partial \Phi}{\partial I_1} + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\frac{\partial \Phi}{\partial I_2}\right)\lambda_i{}^2 - 2\frac{\partial \Phi}{\partial I_2}\,\lambda_i^4 \quad i = 1, 2, 3}$$

With the conditions $J = \frac{dv}{dV} = \lambda_1\lambda_2\lambda_3 = 1 \qquad I_3 = \lambda_1{}^2\lambda_2{}^2\lambda_3{}^2 = 1$

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

Forms of Strain Energy Functions

The constitutive equation is specified once the energy function is known.

The mathematical conditions imposed until now are based on objectivity and isotropy.

Other requirements can come from the type of boundary value problem, the experimental configuration, and the uniqueness of the solution.

In general, the explicit definition of the energy function is based on methodological developments, experimental data, and/or the material microstructure.

For incompressible material $I_3 = {\lambda_1}^2 {\lambda_2}^2 {\lambda_3}^2 = 1$ the energy function is expressed as follows

$$\boxed{\Phi(I_1, I_2) = \sum_{i,j=0}^{\infty} C_{ij} (I_1 - 3)^i (I_2 - 3)^j \,.}$$

In practice a small number of terms is required.

The material parameters are calculated by detailed experimentation and carful identification.

The larger the number of terms the process becomes more difficult.

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

Forms of Strain Energy Functions

Two simple forms of the energy function

$$\Phi(I_1, I_2) = \sum_{i,j=0}^{\infty} C_{ij}(I_1 - 3)^i (I_2 - 3)^j \, .$$

Mooney-Rivlin Strain Energy Function

$$\Phi(I_1, I_2) = C_{10}(I_1 - 3) + C_{01}(I_2 - 3)$$

Important in the development of non-linear elasticity
(good for stretch ratios up to 4)

Neo-Hookean model

$$\Phi(I_1) = C_{10}(I_1 - 3)$$

The model has its origin in the statistical theory of rubber elasticity.
(good for stretch ratios less than 2)

$$C_{10} = nk_BT$$

n: number of polymer chains per unit volume
k_B : Boltzmann's constant
T : absolute temperature

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

Example: simple stress states

1:Biaxial stretch

We have two independent stretches $\lambda_1, \ \lambda_2$

Fro the incompressibility condition

$$J = \frac{dv}{dV} = \lambda_1 \lambda_2 \lambda_3 = 1$$

$$\lambda_3 = \lambda_1^{-1} \lambda_2^{-1}$$

Stresses $\sigma_1, \, \sigma_2 \neq 0$, $\sigma_3 = 0$

Introduce $\sigma_3 = 0$ in

$$\sigma_i = -p + 2 \left(\frac{\partial \Phi}{\partial I_1} + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \frac{\partial \Phi}{\partial I_2} \right) \lambda_i{}^2 - 2 \frac{\partial \Phi}{\partial I_2} \lambda_i^4$$

$$p = 2 \frac{1}{\lambda_1^2 \lambda_2^2} \frac{\partial \Phi}{\partial I_1} + 2 \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 \lambda_2^2} \frac{\partial \Phi}{\partial I_2}$$

$$\begin{aligned} \sigma_1 = & \quad 2 \left(\lambda_1^2 - \frac{1}{\lambda_1^2 \lambda_2^2} \right) \left(\frac{\partial \Phi}{\partial I_1} + \lambda_2^2 \frac{\partial \Phi}{\partial I_2} \right) \\ \sigma_2 = & \quad 2 \left(\lambda_2^2 - \frac{1}{\lambda_1^2 \lambda_2^2} \right) \left(\frac{\partial \Phi}{\partial I_1} + \lambda_1^2 \frac{\partial \Phi}{\partial I_2} \right) . \end{aligned}$$

Incompressible Hyperelastic Isotropic Materials

Example: simple stress states

2: Equibiaxial Stretch

We have $\sigma_1 = \sigma_2 = \sigma \, , \sigma_3 = 0.$

and $\lambda_1 = \lambda_2 = \lambda$

$$\begin{aligned} \sigma_1 &= \quad 2\left(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial\Phi}{\partial I_1} + \lambda_2^2\frac{\partial\Phi}{\partial I_2}\right) \\ \sigma_2 &= \; 2\left(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial\Phi}{\partial I_1} + \lambda_1^2\frac{\partial\Phi}{\partial I_2}\right). \end{aligned}$$

$$\sigma = 2\left(\lambda^2 - \frac{1}{\lambda^4}\right)\left(\frac{\partial\Phi}{\partial I_1} + \lambda^2\frac{\partial\Phi}{\partial I_2}\right).$$

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

Example: simple stress states

3: Uniaxial Stretch

We have $\lambda_1 = \lambda$, and $\lambda_2 = \lambda_3 = \lambda^{-1/2}$.

$\sigma_1 = \sigma$, $\sigma_2 = \sigma_3 = 0$

$$\boxed{\begin{aligned} \sigma_1 = &\quad 2\left(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial\Phi}{\partial I_1} + \lambda_2^2\frac{\partial\Phi}{\partial I_2}\right) \\ \sigma_2 = &\; 2\left(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial\Phi}{\partial I_1} + \lambda_1^2\frac{\partial\Phi}{\partial I_2}\right). \end{aligned}}$$

$$\boxed{\sigma = 2\left(\lambda^2 - \frac{1}{\lambda}\right)\left(\frac{\partial\Phi}{\partial I_1} + \frac{1}{\lambda}\frac{\partial\Phi}{\partial I_2}\right)}$$

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

Example: Inflation of a Balloon

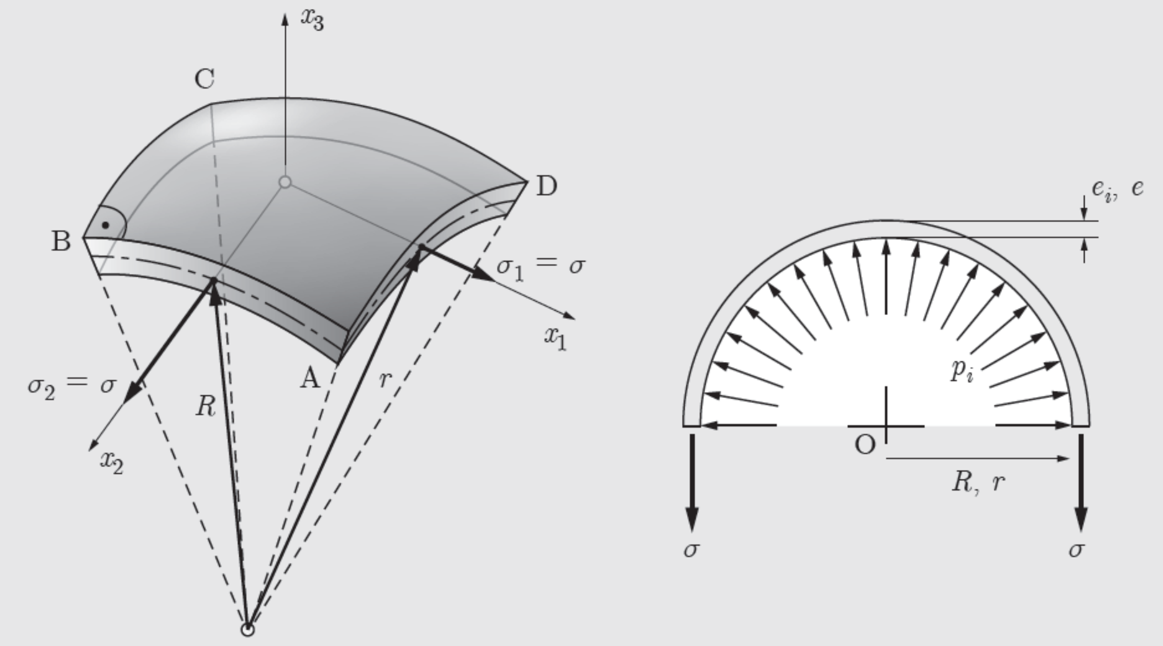


Fig. 6.3 Inflation of a balloon: (a) geometry and (b) boundary conditions

Based on the spherical symmetry we have

$$\sigma_1 = \sigma_2 = \sigma, \sigma_3 = 0.$$

From equilibrium

$$\pi r^2 p_i = 2\pi r e\sigma \longrightarrow p_i = 2\frac{e}{r}\sigma$$

Stretch ratio

$$\lambda = r/R$$

Incompressibility condition

$$4\pi r^2 e = 4\pi R^2 e_i \longrightarrow \frac{e}{e_i} = \frac{1}{\lambda^2}$$

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

Example: Inflation of a Balloon

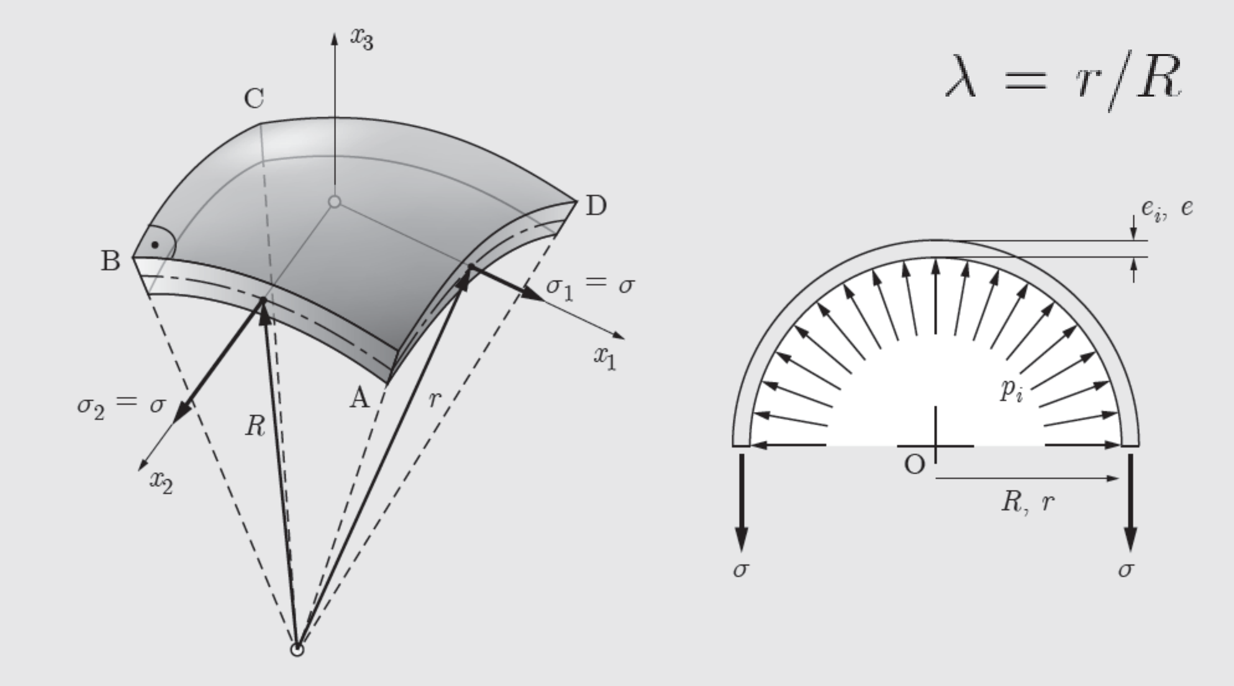


Fig. 6.3 Inflation of a balloon: (a) geometry and (b) boundary conditions

Neo-Hookean model

Stresses

$$\sigma = 2\left(\lambda^2 - \frac{1}{\lambda^4}\right)\frac{\partial\Phi}{\partial I_1} = 2C_{10}\left(\lambda^2 - \frac{1}{\lambda^4}\right)$$

Pressure

$$p_i = 2\frac{e}{r}\sigma \rightarrow p_i(\lambda) = 4C_{10}\frac{e_i}{R}\frac{1}{\lambda}\left(1 - \frac{1}{\lambda^6}\right)$$

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

Example: Inflation of a Balloon

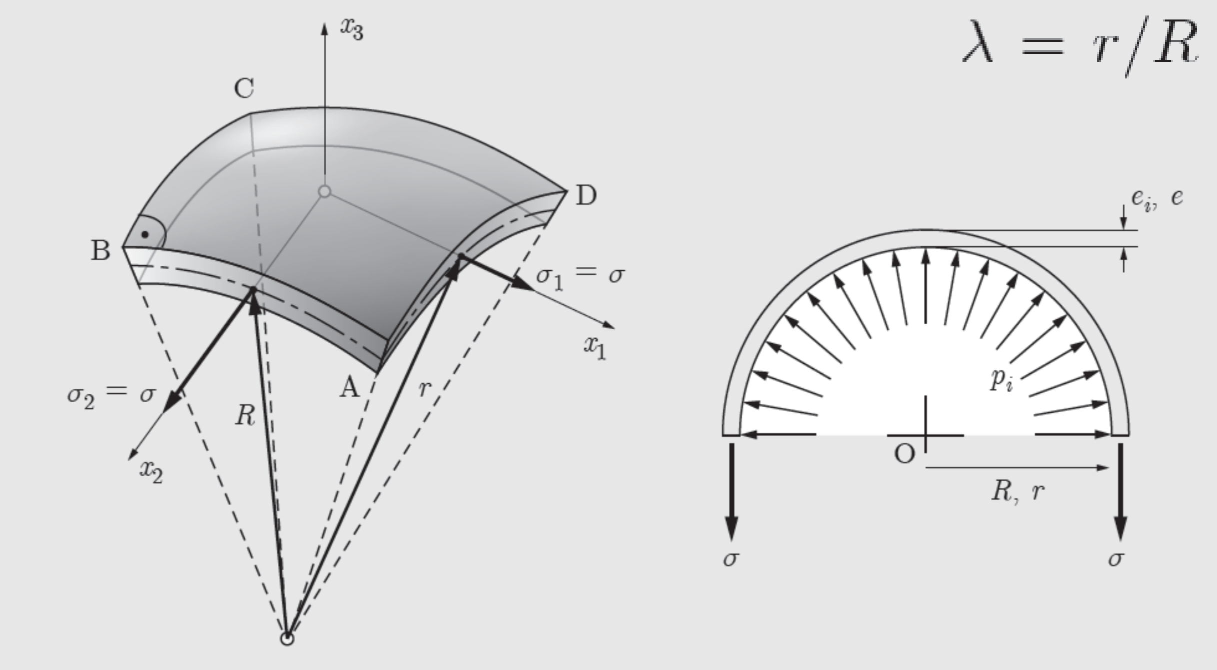


Fig. 6.3 Inflation of a balloon: (a) geometry and (b) boundary conditions

Mooney-Rivlin model

$$C_{10} = \partial\Phi/\partial I_1$$
$$C_{01} = \partial\Phi/\partial I_2$$

Stresses

$$\sigma = 2\left(\lambda^2 - \frac{1}{\lambda^4}\right)\left(C_{10} + \lambda^2 C_{01}\right)$$

Pressure

$$p_i = 2\frac{e}{r}\sigma$$

$$p_i(\lambda) = 4C_{10}\frac{e_i}{R}\frac{1}{\lambda}\left(1 - \frac{1}{\lambda^6}\right)\left(1 + \eta\lambda^2\right)$$

with $\eta = C_{01}/C_{10}$

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

Example: Inflation of a Balloon

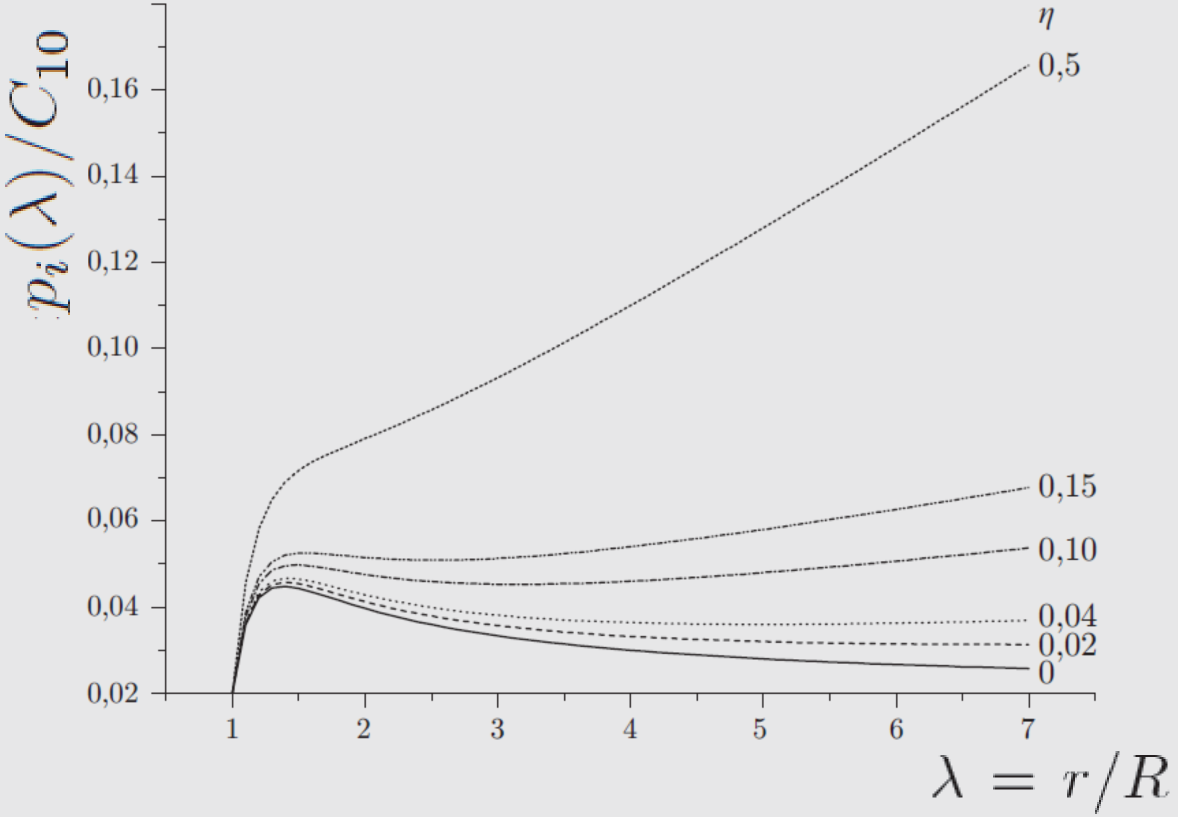


Fig. 6.4 Normalized pressure in a balloon as a function of different values of η

Mooney-Rivlin Model

$$p_i(\lambda) = 4C_{10}\frac{e_i}{R}\frac{1}{\lambda}\left(1-\frac{1}{\lambda^6}\right)(1+\eta\lambda^2)$$

Neo-Hookean Model

$$p_i(\lambda) = 4C_{10}\frac{e_i}{R}\frac{1}{\lambda}\left(1-\frac{1}{\lambda^6}\right)$$

Neo-Hookean Model $\eta = 0$

Solid Mechanics: Non-linear elasticity

Incompressible Hyperelastic Isotropic Materials

Forms of Strain Energy Functions

Ogden's Model

$$\phi(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i} \left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\right)$$

α_i and μ_i are constants obtained
From experimental data and identification

This model gives very good results for when *N*=3 (or higher).

It reduces to Neo-Hookean with

$$N = 1, \alpha_1 = 2, C_{10} = \mu_1/2$$

We obtain the Moonley-Rivlin with

$$N = 2, \alpha_1 = 2, \alpha_2 = -2,$$

$$C_{10} = \mu_1/2, C_{01} = -\mu_2/2$$